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Exactly linearizable maps and $SU(n)$ coherent states

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Abstract. Classical linear maps associated with quantum maps are investigated. It is demonstrated that with the help of $SU(n)$ coherent states, $SU(n)$ tensors fulfilling nonlinear identities can be constructed. Nonlinear maps which evolve like linear maps if initial conditions lie on a manifold which is explicitly given are defined and the analogy with the Kolmogorov–Arnold–Moser theorem is discussed. The problem of geometric quantization is also investigated and a geometric relation between the analysed quantum and classical maps is found.

1. Introduction

The purpose of this paper is to construct and investigate nonlinear dissipative maps which for some initial conditions evolve like linear maps. In this paper we shall study classical maps generated by quantum maps. Let us thus consider the unitary one-step evolution of operators Y_i in the Heisenberg picture:

$$Y_i(n+1) = U^\dagger Y_i(n) U \tag{1}$$

$$U = \exp(-\mathrm{i}H) \tag{2}$$

where H is constructed as a polynomial in operator Y_i-elements of some finite-dimensional Lie algebra $\mathfrak{a}$:

$$H = \sum c_{ij\ldots k}^{pq\ldots r}\, Y_i^p Y_j^q \ldots Y_k^r. \tag{3}$$

A class of systems obtained for $\mathfrak{a} = \mathfrak{su}(2)$ is the so-called quantum kicked top model, investigated in connection with quantum-classical correspondence for classically chaotic systems [1, 2]. Let us consider the concrete example of such a quantum map where $Y_i = J_i \in \mathfrak{su}(2)$, $i = 1, 2, 3$:

$$U = U_1 U_2 = \exp\left(-\mathrm{i}\frac{k}{2j+1} J_1^2\right) \exp(-\mathrm{i}pJ_3) = \exp(-\mathrm{i}H). \tag{4}$$

More exactly, J_i are infinitesimal generators of rotations in the $(2j+1)$-dimensional representation of spin j, where $\boldsymbol{J}^2 = (J_1)^2 + (J_2)^2 + (J_3)^2 = j(j+1)\mathbf{1}$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $\mathbf{1}$ is the $(2j+1) \times (2j+1)$ identity matrix. In equation (4) k and p are parameters [1–3] and we have performed for further convenience a cyclic permutation of components of $\boldsymbol{J}$ with respect to the formula for U in [1].

The nonlinear equations of motion for components of $\boldsymbol{J}$, equations (1) and (4),

$$\boldsymbol{J}' = \mathrm{e}^{\mathrm{i}pJ_3} \mathrm{e}^{\mathrm{i}\kappa J_1^2} \boldsymbol{J} \mathrm{e}^{-\mathrm{i}\kappa J_1^2} \mathrm{e}^{-\mathrm{i}pJ_3} \tag{5}$$

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result in

$$
\begin{aligned}
J_1' &= J_1 C_p - J_2 S_p \\
J_2' &= \tfrac{1}{2}(J_2 C_p + J_1 S_p + \mathrm{i} J_3)\mathrm{e}^{\mathrm{i}2\kappa(J_1 C_p - J_2 S_p + \frac{1}{2})} + \text{h.c.} \\
J_3' &= \frac{1}{2\mathrm{i}}(J_2 C_p + J_1 S_p + \mathrm{i} J_3)\mathrm{e}^{\mathrm{i}2\kappa(J_1 C_p - J_2 S_p + \frac{1}{2})} + \text{h.c.}
\end{aligned}
\tag{6}
$$

where $\kappa = \frac{k}{2j+1}$, $C_p \equiv \cos(p)$, $S_p \equiv \sin(p)$ [1]. The most important features of the model are the finite dimensionality of its Hilbert space and the capability of chaotic motion in the classical limit. This model is well suited to study the transition between quantum dynamics (which can be linearized due to the finite dimensionality of the corresponding Hilbert space [3]) and nonlinear classical dynamics by increasing to infinity the dimensionality of the quantum system [1, 2].

In the particular case of the quantum kicked top, equation (4), the generators J_i are thus represented by $(2j+1) \times (2j+1)$ complex matrices. The same is true for the elements of the universal covering algebra generated by polynomials in J_i (as, for example, in equation (3)). Hence we achieve in this way an embedding of the system in $\mathfrak{u}(2j+1)$ algebra (or, in fact $\mathfrak{su}(2j+1)$ since the remaining generator of $\mathfrak{u}(2j+1)$ is proportional to the $(2j+1) \times (2j+1)$ identity matrix and its evolution is trivial).

Extending the Heisenberg equations of motion to m generators X_i of $\mathfrak{su}(2j+1)$ algebra, $m = (2j+1)^2 - 1$,

$$
X_i(n+1) = U^\dagger X_i(n) U \tag{7}
$$

it is possible to linearize equation (7) exactly:

$$
X_i(n+1) = \sum_{k=1}^{m} A_{ik} X_k(n) \tag{8}
$$

where $i = 1, \ldots, m$, with orthogonal matrix A_{ik}, i.e. $A_{ki} = (A^{-1})_{ik}$. Moreover, H defined by (4) belongs due to the Hausdorff–Baker–Campbell theorem to a subalgebra spanned by J_3, J_1^2 and all their commutators, i.e. $H \in (J_3, J_1^2)_j \subset \mathfrak{u}(2j+1)$ [3].

We shall describe in the next section how isomorphic linear parameter dynamics can be associated formally with operator dynamics (8):

$$
c_i(n+1) = \sum_{k=1}^{m} A_{ki} c_k(n). \tag{9}
$$

This leads to the question of connection between quantum and classical maps, cf equations (8) and (9), respectively, which we shall investigate in section 5 in the context of geometric quantization [4].

Let us now consider for the sake of a simple example a linear map:

$$
\begin{aligned}
c_1(n+1) &= c_1(n)\cos\alpha + c_2(n)\sin\alpha \\
c_2(n+1) &= -c_1(n)\sin\alpha + c_2(n)\cos\alpha
\end{aligned}
\tag{10}
$$

which can be obtained from a quantum map involving the generator of the $\mathfrak{o}(2)$ algebra (this map is a special case of the map discussed in section 4, equation (25)).

Obviously, the map (10) has the $\mathfrak{o}(2)$ invariant: $(c_1(n))^2 + (c_2(n))^2$. Let us perturb the map (10):

$$
\begin{aligned}
c_1(n+1) &= c_1(n)[(1-\varepsilon) + \varepsilon(c_1^2(n) + c_2^2(n))]\cos\alpha + c_2(n)\sin\alpha \\
c_2(n+1) &= -c_1(n)\sin\alpha + c_2(n)\cos\alpha.
\end{aligned}
\tag{11}
$$

The obtained map is nonlinear, yet for initial conditions fulfilling $c_1^2(0)+c_2^2(0) = 1$ the dynamics is linear (provided that the motion on the unit circle is stable). We shall demonstrate that the Lie algebraic operator formulation (7) provides the framework to construct more general nonlinear dynamical systems, the dynamics of which is linear on some manifold, which will be referred to as linear manifold $\mathcal{L}$.

The paper is organized as follows. In section 2 we describe how parameter dynamics can be induced by a quantum map acting in $\mathfrak{su}(2j+1)$ algebra. In section 3 properties of $SU(2j+1)$ tensors obtained by computing averages of $\mathfrak{su}(2j+1)$ generators over $SU(2j+1)$ coherent states [5, 6] are described. Applications to nonlinear maps are discussed in the next section—nonlinear maps which evolve like linear maps for initial conditions on manifold $\mathcal{L}$ which is explicitly given are constructed and results of numerical simulations are described. In section 5 the problem of geometric quantization is investigated and a geometric relation between the quantum map (4) and the corresponding classical map is found. In the last section the results are compared with the Kolmogorov–Arnold–Moser (KAM) theorem [7–9].

2. Parameter dynamics induced by quantum maps

The extended operator equation of motion (7), operating in $\mathfrak{su}(2j+1)$ Lie algebra, yields the possibility of defining a dual linear dynamics in parameter space [3, 10]. Let us consider a unitary transformation of a linear combination of $\mathfrak{su}(2j+1)$ generators X_k:

$$\sum_{k=1}^{m} c_k X'_k = U^\dagger \left(\sum_{k=1}^{m} c_k X_k \right) U. \tag{12}$$

Since the parameters c_k are arbitrary, equation (12) is equivalent to (7). In this picture operators evolve and the parameters are fixed. It follows that the generators evolve according to (8) where the matrix A is known. Alternatively, the parameters evolve while the operators are fixed:

$$\sum_{k=1}^{m} c'_k X_k = U^\dagger \left(\sum_{k=1}^{m} c_k X_k \right) U. \tag{13}$$

Using a representation of generators orthogonal in the scalar product [3, 11]

$$\tfrac{1}{2}\,\mathrm{Tr}(X_i X_j) = \delta_{ij} \tag{14}$$

we obtain equation (9) describing evolution of the parameters:

$$c'_i = \tfrac{1}{2}\,\mathrm{Tr}\left(U^\dagger \left(\sum_{k=1}^{m} c_k X_k \right) U X_i \right) = \sum_{k=1}^{m} \tfrac{1}{2}\,\mathrm{Tr}(U^\dagger X_k U X_i) c_k = \sum_{k=1}^{m} A_{ki} c_k. \tag{15}$$

Since the matrix A is orthogonal, $A_{ik} = (A_{ki})^{-1}$, the evolution equations (8) and (15) differ in the time direction.

3. $SU(n)$ coherent states and averages of $SU(n)$ generators

We shall base our approach on methods described in [12] for $SU(3)$ Perelomov coherent states. To construct Perelomov states in the case of the $SU(N)$ group we consider the complex extension $\mathfrak{sl}(n, \mathbb{C})$ of the Lie algebra $\mathfrak{su}(n)$ spanned by the generators S_{ij} [5]:

$$[S_{ij}, S_{kl}] = \delta_{kj} S_{il} - \delta_{il} S_{kj}$$
$$S_{ij}^\dagger = S_{ji}$$

and commuting operators H_i:

$$H_i = \tfrac{1}{2}(S_{ii} - S_{i+1,i+1}) \qquad i = 1, 2, \ldots, n-1$$

represented irreducibly by $m \times m$ complex matrices acting in $\mathbb{C}^m$. Let $|\mu\rangle$ be the highest-weight vector of the representation, i.e.

$$S_{ij}|\mu\rangle = 0 \qquad i < j.$$

Then a coherent state is defined as

$$|\gamma\rangle = \frac{\|\gamma\rangle}{\langle\gamma\|\gamma\rangle^{1/2}}$$

$$\|\gamma\rangle = \exp\left(\sum_{i>j} \gamma_{ij} S_{ij}\right)|\mu\rangle.$$

It is easy to show that the normalized coherent state $|\gamma\rangle$ is thus obtained by acting on the highest-weight vector by an unitary operator U representing some element of the group $SU(n)$.

In the canonical basis $|i\rangle$, $i = 1, \ldots, n$ in $\mathbb{C}^n$ where $|i\rangle$ is the column vector with one on ith position and zeros elsewhere, $|i\rangle = \langle i|^{\mathrm{T}}$, the generators are represented as $S_{ij} = |i\rangle\langle j|$, and the highest-weight vector reads

$$|\mu\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{16}$$

We check that $H_1|\mu\rangle = 1$, $H_i|\mu\rangle = 0$, $i > 1$ so the representation is degenerate.

It follows easily that

$$\exp\left(\sum_{i>j} \gamma_{ij} S_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \gamma_1 & 1 & 0 & \cdots & 0 \\ \gamma_2 & \gamma_n & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & 1 & 0 \\ \gamma_{n-1} & \gamma_{2n-3} & \gamma_{3n-6} & \gamma_{\frac{n(n-1)}{2}} & 1 \end{pmatrix} \equiv b_-[\gamma] \tag{17}$$

i.e. the element b_- belongs to the subgroup of lower triangular matrices in $SL(n, \mathbb{C})$ having unit diagonal elements and is parametrized by $n(n-1)$ complex numbers $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_{\frac{n(n-1)}{2}}]$, which are linear combinations of γ_{ij} (we assume parametrization of b_- in terms of γ and hence we do not need to know the explicit dependence of b_- on γ_{ij}).

We can define the unnormalized coherent state as

$$\|\gamma\rangle = b_-[\gamma]|\mu\rangle = \begin{pmatrix} 1 \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{n-1} \end{pmatrix} \tag{18}$$

so that the normalized coherent state reads

$$|\gamma\rangle = \frac{1}{\sqrt{1 + \gamma_1\gamma_1^* + \cdots + \gamma_{n-1}\gamma_{n-1}^*}}\|\gamma\rangle. \tag{19}$$

Let us consider generators X_i spanning the $(J_3, J_1^2)_j^\perp$ space, i.e. the orthogonal complement to $\mathfrak{u}(n)$, $n = 2j + 1$, in the scalar product (14). We can obtain the so-called momentum representation [13] of $SU(2j + 1)$ tensor T_i^α in this subspace in the form $\sum_k \langle\gamma|X_k|\gamma\rangle X_k$ where we sum over all $X_k \in (J_3, J_1^2)_j^\perp$.

We obtain, up to a normalization constant N_n,

$$\hat{T}_i^\alpha = PT_i^\alpha = N_n \begin{pmatrix} 0 & \gamma_1^* & 0 & \gamma_3^* & \cdots & 0 & \gamma_{n-1}^* \\ \gamma_1 & 0 & \gamma_1\gamma_2^* & 0 & \cdots & \gamma_1\gamma_{n-2}^* & 0 \\ 0 & \gamma_2\gamma_1^* & 0 & \gamma_2\gamma_3^* & \cdots & 0 & \gamma_2\gamma_{n-1}^* \\ \gamma_3 & 0 & \gamma_3\gamma_2^* & 0 & \cdots & \gamma_3\gamma_{n-2}^* & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \gamma_{n-2}\gamma_1^* & 0 & \gamma_{n-2}\gamma_3^* & \cdots & 0 & \gamma_{n-2}\gamma_{n-1}^* \\ \gamma_{n-1} & 0 & \gamma_{n-1}\gamma_2^* & 0 & \cdots & \gamma_{n-1}\gamma_{n-2}^* & 0 \end{pmatrix} \tag{20}$$

where P denotes projection onto $(J_3, J_1^2)_j^\perp$ and we have used the standard representation of generators J_i with diagonal J_3.

Theorem 1. *Let $SU(2j+1)$ tensor T_i^α be constructed in the following way: $T_i^\alpha = \sum_k X_k\langle\gamma|X_k|\gamma\rangle$ where averages $\langle\gamma|X_k|\gamma\rangle$ are computed over a (degenerate) representation of $SU(2j+1)$ coherent states. Let $\hat{T}_i^\alpha$ be the projection of T_i^α onto $(J_3, J_1^2)_j^\perp$. Then the tensor $\hat{T}_i^\alpha$ fulfills the identity $\hat{T}^3 = \hat{T}$.*

Proof. Let us consider the form (20). We check directly that the choice of the normalization constant N_n

$$N_n = \frac{1}{\sqrt{(1+\gamma_2\gamma_2^* + \cdots + \gamma_{n-2}\gamma_{n-2}^*)(\gamma_1\gamma_1^* + \cdots + \gamma_{n-1}\gamma_{n-1}^*)}} \tag{21}$$

leads to the demanded property $\hat{T}^3 = \hat{T}$ and hence the theorem follows. □

We should mention here that the assumption that the representation of coherent states is degenerate is crucial because the theorem 1 is not valid for general representation. This can be readily verified in the case of $SU(3)$ coherent states using results of [12]. Moreover, performing similar computations we arrive at an analogous result:

$$\check{T} \overset{\mathrm{df}}{=} \hat{T}^2 \in (J_3, J_1^2)_j. \tag{22}$$

4. An example of an exactly linearizable map

4.1. Linear map

Let us consider a matrix $M = \sum_k c_k X_k \in \mathfrak{su}(4)$:

$$M = \begin{pmatrix} 0 & c_1 - \mathrm{i}c_2 & 0 & c_3 - \mathrm{i}c_4 \\ c_1 + \mathrm{i}c_2 & 0 & c_5 - \mathrm{i}c_6 & 0 \\ 0 & c_5 + \mathrm{i}c_6 & 0 & c_7 - \mathrm{i}c_8 \\ c_3 + \mathrm{i}c_4 & 0 & c_7 + \mathrm{i}c_8 & 0 \end{pmatrix} \tag{23}$$

which has the structure of tensor $\hat{T}_i^\alpha$, equation (20). It follows that $M \in (J_3, J_1^2)_{\frac{3}{2}}^\perp$, i.e. M belongs to the orthogonal complement of $(J_3, J_1^2)_{\frac{3}{2}}$—the algebra generated by J_3, J_1^2 and all their commutators—in $\mathfrak{u}(4)$. Operators J_1, J_2, J_3 are generators of $\mathfrak{o}(3)$ algebra for $j = \frac{3}{2}$ and we have used the standard representation of generators J_i with $J_3 = \mathrm{diag}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$.

Let us perform a unitary transformation: $M \to M' = U^\dagger M U$, where we substitute $U = \mathrm{e}^{-\mathrm{i}\alpha J_3}$. We shall treat the transformation of M' as transformation of parameters $c_1, c_2, \ldots, c_8$ with $\mathfrak{su}(4)$ generators spanning $(J_3, J_1^2)_{\frac{3}{2}}^\perp$ unchanged (cf equation (13)):

$$M' = U^\dagger M U = \sum_k c_k' X_k. \tag{24}$$

This linear map $\boldsymbol{c}' = A\boldsymbol{c}$, $\boldsymbol{c} = [c_1, c_2, \ldots, c_8]^{\mathrm{T}}$, where $^{\mathrm{T}}$ denotes the transposition of a matrix, is easily computed, cf equation (15):

$$\begin{aligned} c_i' &= c_i \cos\lambda\alpha + c_j \sin\lambda\alpha \\ c_j &= -c_i \sin\lambda\alpha + c_j \cos\lambda\alpha \end{aligned} \tag{25}$$

where $\lambda = 1$ for $(i, j) = (1, 2), (5, 6), (7, 8)$ and $\lambda = 3$ for $(i, j) = (3, 4)$.

4.2. Nonlinear map

Let us define $N = M^3$ with M given by equation (23):

$$N = M^3 = \begin{pmatrix} 0 & d_1 - \mathrm{i}d_2 & 0 & d_3 - \mathrm{i}d_4 \\ d_1 + \mathrm{i}d_2 & 0 & d_5 - \mathrm{i}d_6 & 0 \\ 0 & d_5 + \mathrm{i}d_6 & 0 & d_7 - \mathrm{i}d_8 \\ d_3 + \mathrm{i}d_4 & 0 & d_7 + \mathrm{i}d_8 & 0 \end{pmatrix} \in (J_3, J_1^2)^{\perp}_{\frac{3}{2}}. \tag{26}$$

$N = \sum_k d_k X_k \in \mathfrak{su}(4)$ and the parameters of N are easily represented in terms of parameters of M:

$$\begin{aligned}
d_1 &= c_1(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2) + c_3c_7c_5 + c_4c_8c_5 - c_3c_8c_6 + c_4c_7c_6 \\
d_2 &= c_2(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2) - c_6c_7c_3 - c_5c_8c_3 + c_5c_7c_4 - c_6c_8c_4 \\
d_3 &= c_3(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_7^2 + c_8^2) + c_1c_5c_7 - c_2c_6c_7 - c_1c_6c_8 - c_2c_5c_8 \\
d_4 &= c_4(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_7^2 + c_8^2) + c_8c_5c_1 + c_7c_6c_1 + c_7c_5c_2 - c_8c_6c_2 \\
d_5 &= c_5(c_1^2 + c_2^2 + c_5^2 + c_6^2 + c_7^2 + c_8^2) - c_2c_3c_8 + c_1c_3c_7 + c_2c_4c_7 + c_1c_4c_8 \\
d_6 &= c_6(c_1^2 + c_2^2 + c_5^2 + c_6^2 + c_7^2 + c_8^2) - c_8c_3c_1 + c_7c_4c_1 - c_7c_3c_2 - c_8c_4c_2 \\
d_7 &= c_7(c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2 + c_8^2) - c_2c_6c_3 + c_2c_5c_4 + c_1c_6c_4 + c_4^2c_7 \\
d_8 &= c_8(c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2 + c_8^2) - c_3c_6c_1 - c_4c_6c_2 + c_4c_5c_1 - c_3c_5c_2.
\end{aligned} \tag{27}$$

Let us define another map $N \to N' = U^\dagger N U$:

$$N' = U^\dagger N U = \sum_k d_k' X_k. \tag{28}$$

The corresponding linear map reads

$$\boldsymbol{d}' = A\boldsymbol{d} \qquad \boldsymbol{d} = [d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8]^{\mathrm{T}} \tag{29}$$

with matrix A the same as in equation (25).

The map $N \to N' = U^\dagger N U$ is isomorphic to the map $M \to M' = U^\dagger M U$. Indeed, $N' = U^\dagger N U = U^\dagger M^3 U = (U^\dagger M U)(U^\dagger M U)(U^\dagger M U) = (M')^3$. It thus follows that transformation of tensor N, a nonlinear representation of the tensor M, is induced by transformation of the tensor M.

Let us now substitute c_1 in the first of equations (25) by $d_1 = d_1(c_1, \ldots, c_8)$, equations (27), to obtain a nonlinear map:

$$\begin{aligned}
c_1' &= \{\varepsilon[c_1(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2) + c_3c_7c_5 + c_4c_8c_5 - c_3c_8c_6 + c_4c_7c_6] \\
&\qquad + (1 - \varepsilon)c_1\} \cos\alpha + c_2 \sin\alpha \\
c_2' &= -c_1 \sin\alpha + c_2 \cos\alpha \\
c_3' &= c_3 \cos 3\alpha + c_4 \sin 3\alpha \\
c_4' &= -c_3 \sin 3\alpha + c_4 \cos 3\alpha \\
c_5' &= c_5 \cos\alpha + c_6 \sin\alpha \\
c_6' &= -c_5 \sin\alpha + c_6 \cos\alpha \\
c_7' &= c_7 \cos\alpha + c_8 \sin\alpha \\
c_8' &= -c_7 \sin\alpha + c_8 \cos\alpha
\end{aligned} \tag{30}$$

i.e. for $\varepsilon = 0$ we recover the unperturbed map, while for $\varepsilon = 1$ variable c_1 is substituted by d_1, cf equations (27). For initial condition $[c_1(0), c_2(0), 0, 0, 0, 0, 0, 0]^{\mathrm{T}}$ the map (30) is equivalent to the map (11).

The initial map (25) and the perturbation were so designed that equations for variables $c_3, c_4, \ldots, c_8$ could be easily solved and the solutions can be substituted into the first two equations (30) to yield a two-dimensional non-autonomous map.

To linearize the map (30) we identify vectors $\boldsymbol{c}$ and $\boldsymbol{d}$. We thus put in equations (27)

$$d_1 = c_1, d_2 = c_2, \ldots, d_8 = c_8. \tag{31}$$

Let us substitute into the initial condition of equations (30), $\boldsymbol{c}_0$, a solution of equations (27), (31). It follows that since the maps (25) and (29) are equivalent and they have the same initial conditions the map (30) becomes a linear map.

To find non-zero solutions of equations (27) and (31) the theorem 1 is used. The solution is $M = \hat{T}$, i.e. real parameters $c_1, c_2, \ldots, c_8$ are given by

$$\begin{pmatrix} 0 & c_1 - \mathrm{i}c_2 & 0 & c_3 - \mathrm{i}c_4 \\ c_1 + \mathrm{i}c_2 & 0 & c_5 - \mathrm{i}c_6 & 0 \\ 0 & c_5 + \mathrm{i}c_6 & 0 & c_7 - \mathrm{i}c_8 \\ c_3 + \mathrm{i}c_4 & 0 & c_7 + \mathrm{i}c_8 & 0 \end{pmatrix} = \frac{1}{\sqrt{(1+\gamma_2\gamma_2^*)(\gamma_1\gamma_1^* + \gamma_3\gamma_3^*)}} \times \begin{pmatrix} 0 & \gamma_1^* & 0 & \gamma_3^* \\ \gamma_1 & 0 & \gamma_1\gamma_2^* & 0 \\ 0 & \gamma_2\gamma_1^* & 0 & \gamma_2\gamma_3^* \\ \gamma_3 & 0 & \gamma_3\gamma_2^* & 0 \end{pmatrix} \tag{32}$$

where $\gamma_1, \gamma_2, \gamma_3$ are arbitrary complex numbers. If we set $\gamma_1 = r_1\mathrm{e}^{\mathrm{i}\beta_1}$, $\gamma_2 = r_2\mathrm{e}^{\mathrm{i}\beta_2}$ and $\gamma_3 = r_3\mathrm{e}^{\mathrm{i}\beta_3}$ we obtain $c_1 + \mathrm{i}c_2 = \lambda r_1\mathrm{e}^{\mathrm{i}\beta_1}$, $c_3 + \mathrm{i}c_4 = \lambda r_3\mathrm{e}^{\mathrm{i}\beta_3}$, $c_5 + \mathrm{i}c_6 = \lambda r_1 r_2\mathrm{e}^{\mathrm{i}(\beta_2-\beta_1)}$, $c_7 + \mathrm{i}c_8 = \lambda r_2 r_3\mathrm{e}^{\mathrm{i}(\beta_3-\beta_2)}$, $\lambda = 1/\sqrt{(1+r_2^2)(r_1^2+r_3^2)}$. This solution gives an explicit formula for the linear manifold $\mathcal{L}$.

4.3. Computational results

We have first performed computations with initial conditions on $\mathcal{L}$. We shall perform computations for $\gamma_1 = \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi/4}$, $\gamma_2 = \mathrm{e}^{\mathrm{i}\pi/4}$ and $\gamma_3 = \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi/4}$, i.e. for the initial condition $\boldsymbol{c}_0 = \frac{\sqrt{2}}{4}(1, 1, -1, 1, 1, 1, 1, 1)$.

Since the equations for $c_3, c_4, c_5, c_6, c_7, c_8$ are linear, equation (30), these variables stay on the manifold $\mathcal{L}$. We can thus look for another solution solving equations (27), (31) for c_1, c_2 only:

$$\begin{aligned} c_1 &= c_1(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2) + c_3c_7c_5 + c_4c_8c_5 - c_3c_8c_6 + c_4c_7c_6 \\ c_2 &= c_2(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2) - c_6c_7c_3 - c_5c_8c_3 + c_5c_7c_4 - c_6c_8c_4 \end{aligned} \tag{33}$$

with other variables fixed on the linear manifold: $-c_3 = c_4 = c_5 = c_6 = c_7 = c_8 = \frac{\sqrt{2}}{4}$. We recover the known solution $(c_1, c_2)^{(1)} = \frac{\sqrt{2}}{4}(1, 1)$ and obtain two new solutions: $(c_1, c_2)^{(\pm)} = \frac{\sqrt{2}}{4}(\mp\frac{\sqrt{5}\pm 1}{2}, \mp\frac{\sqrt{5}\pm 1}{2})$.

There are thus three sets of initial conditions on the linear manifold: $\boldsymbol{c}_0^{(1)} = \frac{\sqrt{2}}{4}(1, 1, \boldsymbol{x}_0)$, $\boldsymbol{c}_0^{(\pm)} = \frac{\sqrt{2}}{4}(\mp\frac{\sqrt{5}\pm 1}{2}, \mp\frac{\sqrt{5}\pm 1}{2}, \boldsymbol{x}_0)$ and $\boldsymbol{x}_0 = (-1, 1, 1, 1, 1, 1)$. In the case of $\boldsymbol{c}_0^{(1)}$, $\boldsymbol{c}_0^{(-)}$ and $\cos\alpha \neq 0$ the motion is unstable while for $\boldsymbol{c}_0^{(+)}$ the motion is stable—the attractor $\mathcal{A}_1$ is thus a small circle of radius $\frac{\sqrt{5}-1}{4}$: $\mathcal{A}_1 = (\frac{\sqrt{5}-1}{4}\cos\beta, \frac{\sqrt{5}-1}{4}\sin\beta)$, $\beta \in [0, 2\pi)$. There are two other attractors, $\mathcal{A}_2$, to be described later, approximately at a distance $r = 2$ from the origin, and

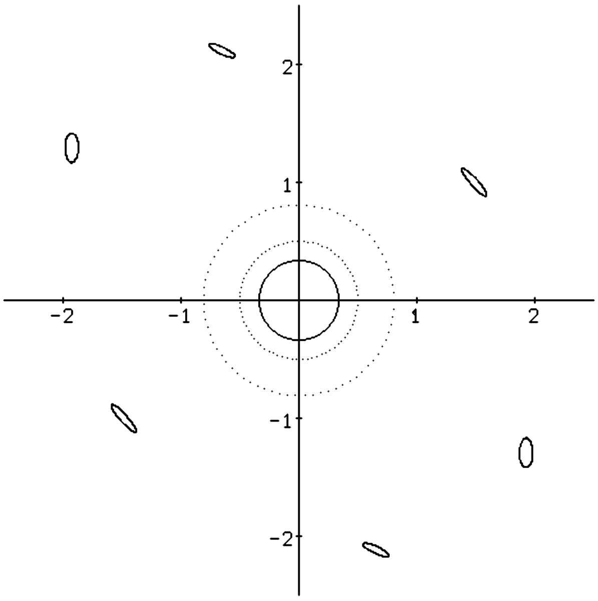

Figure 1. Attractors $\mathcal{A}_1$ and $\mathcal{A}_2$—solid curves, repellers $\mathcal{R}_1$ and $\mathcal{R}_2$—dotted curves; $\cos\alpha = 0.17$, $\varepsilon = 1$.

also an attractor at infinity, $\mathcal{A}_3$. There are also three repellers: $\mathcal{R}_1 = (\frac{\sqrt{5}+1}{4}\cos\beta, \frac{\sqrt{5}+1}{4}\sin\beta)$, $\mathcal{R}_2 = (\frac{1}{2}\cos\beta, \frac{1}{2}\sin\beta)$ and $\mathcal{R}_3 = (0, 0)$. The repellers and the attractor $\mathcal{A}_1$ lie on the linear manifold. In figure 1 the attractors $\mathcal{A}_1$ and $\mathcal{A}_2$ and repellers $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are shown for $\cos\alpha = 0.17$ and $\varepsilon = 1$.

We have also performed computations for more general initial conditions, $\boldsymbol{c}_0^{(1)}(\delta) = \frac{\sqrt{2}}{4}(1, 1, -1, 1, 1 + \delta, 1, 1, 1)$, so that for $\delta = 0$ we obtain the solution of equations (27) and (31) in the form (32), i.e. $\boldsymbol{c}_0^{(1)}(0)$ lies on $\mathcal{L}$. The computations were performed for two sets of control parameters: the angle of rotation α and the perturbation parameter ε—for $\varepsilon = 0$ the system is linear while for $\varepsilon = 1$ parameter c_1 is substituted by d_1 (cf equation (27)). In the first case we have varied $\cos\alpha$ while ε was fixed and equal to unity; in the second case ε was varied and $\cos\alpha$ was held fixed and equal to 0.17. In the case of initial condition $\boldsymbol{c}_0^{(1)}(0)$ the linear motion in the (c_1, c_2) plane is equivalent to rotations along a circle of diameter 0.5. The initial condition was so chosen that the linear motion was unstable, so that the trajectory after leaving the circle had to settle on one of the attractors. To investigate attractors on both sides of the linear manifold $\boldsymbol{c}_0^{(1)}(0)$ (the circle of diameter 0.5) the initial conditions $\boldsymbol{c}_0^{(1)}(\delta)$ with $\delta = \pm 10^{-8}$ were used.

When the parameters are varied the attractor $\mathcal{A}_1$ is unchanged while the attractor $\mathcal{A}_2$ undergoes changes, some of which are described in tables 1 and 2.

5. Inverse problem of geometric quantization

We have demonstrated that linearized equations (8) factorize into dynamics in $(J_3, J_1^2)_j^{\perp}$ and $(J_3, J_1^2)_j$ [3]. Moreover, the dynamics in the subalgebra $(J_3, J_1^2)_j$ is induced by dynamics in the subspace $(J_3, J_1^2)_j^{\perp}$ since equations for generators $X_i \in (J_3, J_1^2)_j$ can be obtained from

Table 1. $\varepsilon = 1$.

$\cos\alpha$	$\delta = -10^{-8}$	$\delta = +10^{-8}$
0.17	$\mathcal{A}_1$	$\mathcal{A}_2$ (six small ovals)
0.192	$\mathcal{A}_1$	$\mathcal{A}_2$ (distorted two-band)
0.193	$\mathcal{A}_1$	$\mathcal{A}_2$ (distorted three-band)
0.194	$\mathcal{A}_1$	$\mathcal{A}_2$ (broad noisy band)
0.195	$\mathcal{A}_1$	$\mathcal{A}_2$ (thin distorted band)
0.196	$\mathcal{A}_1$	$\mathcal{A}_2$ (thin band)
0.199	$\mathcal{A}_1$	$\mathcal{A}_2$ (broad noisy distorted band)

Table 2. $\cos\alpha = 0.17$.

ε	$\delta = -10^{-8}$	$\delta = +10^{-8}$
1	$\mathcal{A}_1$	$\mathcal{A}_2$ (six small ovals)
1.693 34	$\mathcal{A}_1$	$\mathcal{A}_2$ (complicated two-band)
1.70	$\mathcal{A}_1$	$\mathcal{A}_2$ (thin two-band)
1.71	$\mathcal{A}_1$	$\mathcal{A}_2$ (six small ovals)
1.73	$\mathcal{A}_1$	$\mathcal{A}_2$ (very noisy broad band)

dynamical equations for operators $X_k \in (J_3, J_1^2)_j^{\perp}$

$$X_i(n+1) = \sum_{k=p+1}^{m} A_{ik} X_k(n) \tag{34}$$

and their commutation relations [3]. In equation (34) we have divided $\mathfrak{u}(2j+1)$ generators X_i into two subspaces: $\{X_{p+1}, X_{p+2}, \ldots, X_m\} = (J_3, J_1^2)_j^{\perp}$, $\{X_0, X_1, \ldots, X_p\} = (J_3, J_1^2)_j$, $X_0 = \mathbf{1}$, where $m = (2j+1)^2 - 1$, $p = \mathrm{Integer}[\frac{(2j+1)^2}{2}]$.

It is now possible to write down dynamical equations for parameters corresponding to generators $X_i \in (J_3, J_1^2)_j^{\perp}$:

$$c_i(n+1) = \sum_{k=p+1}^{m} A_{ki} c_k(n) \tag{35}$$

using ideas described in section 2, or computing averages over $SU(2j+1)$ coherent states, $c_k(n) \stackrel{\mathrm{df}}{=} \langle\gamma|X_k(n)|\gamma\rangle$ (both methods lead to equivalent equations which differ in the time direction only since the matrix A is orthogonal, $A_{ik} = (A_{ki})^{-1}$).

We face an important problem now if equivalence between the quantum equation (34) and the classical one (35) is to be complete: how to derive dynamical equations for parameters $c_1, \ldots, c_p$ from equation (35) only. This is the inverse problem of geometric quantization [4]: we need additional geometric structure in classical space which would correspond to the Lie algebraic structure of the quantum operators $X_1, X_2, \ldots, X_m$.

To solve this problem let us put $\hat{T} = \sum_{k=p+1}^{m} c_k X_k \in (J_3, J_1^2)_j^{\perp}$ as in equation (20), where we have stripped T_i^{α} of indices for simplicity (see also equation (23)). The map $U^{\dagger}\hat{T}U = \hat{T}' = \sum_{k=p+1}^{m} c_k' X_k$ defines the dynamics of parameters corresponding to generators in $(J_3, J_1^2)_j^{\perp}$, cf equation (35). Let us define a new matrix $\check{T} \stackrel{\mathrm{df}}{=} \hat{T}^2$. Since $\check{T} = \sum_{k=0}^{p} d_k X_k \in (J_3, J_1^2)_j$, cf equation (22), we obtain the map $c_{p+1}, \ldots, c_m \to d_0, d_1, \ldots, d_p$. Moreover, $U^{\dagger}\check{T}U = (U^{\dagger}\hat{T}U)(U^{\dagger}\hat{T}U) = (\sum_{k=p+1}^{m} c_k' X_k)^2 = \sum_{k=0}^{p} d_k' X_k$. It follows that the transformation $U^{\dagger}\check{T}U = \check{T}'$ induces dynamics of new parameters d_k isomorphic to dynamics of $\mathfrak{su}(2j+1)$ generators in $(J_3, J_1^2)_j$.

For example, for $j = 1$ we put $\boldsymbol{J} = [\lambda_7, -\lambda_5, \lambda_2]$ to obtain

$$(J_3, J_1^2)_j^{\perp} = \{X_5, X_6, X_7, X_8\} = \{\lambda_1, \lambda_2, \lambda_6, \lambda_7\}$$

$$\begin{aligned}(J_3, J_1^2)_j &= \{X_0\} \cup \{X_1, X_2, X_3\} \cup \{X_4\} \\ &= \{1\} \cup \left\{\lambda_4, \lambda_5, \frac{1}{2}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8, \right\} \cup \left\{\frac{\sqrt{3}}{2}\lambda_3 - \frac{1}{2}\lambda_8\right\}\end{aligned}$$

where λ are Gell–Mann matrices [14]. The linearized equations of motion read

$$\begin{bmatrix} Z_1^{(N+1)} \\ Z_2^{(N+1)} \end{bmatrix} = \begin{bmatrix} C_p & S_p \\ -S_p e^{-i\frac{k}{3}} & C_p e^{-i\frac{k}{3}} \end{bmatrix} \cdot \begin{bmatrix} Z_1^{(N)} \\ Z_2^{(N)} \end{bmatrix} \tag{36}$$

where $Z_1 = X_5 + iX_6$, $Z_2 = X_7 - iX_8$ and Z_1 and Z_2 commute [3]. Parameters $z_1 = c_5 + ic_6$ and $z_2 = c_7 - ic_8$ evolve according to equation (36). Furthermore, $\hat{T} = \sum_{i=5}^{8} c_i X_i$, $\check{T} \stackrel{\text{df}}{=} \hat{T}^2 = \sum_{i=0}^{4} d_i X$ and it follows that $d_0 = \frac{2}{3}(c_5^2 + c_6^2 + c_7^2 + c_8^2) = -\frac{\sqrt{3}}{4} d_4$ and $(d_1, d_2, d_3) = (c_5c_7 - c_6c_8, c_5c_8 + c_6c_7, \frac{1}{2}(c_5^2 + c_6^2 - c_7^2 - c_8^2))$. Parameters d_1, d_2, d_3 evolve exactly as the operators X_1, X_2, X_3 while d_0, d_4 and X_0, X_4 are constants of motion. The map $(c_5, c_6, c_7, c_8) \to (d_1, d_2, d_3)$ is the Hopf map of the three-dimensional sphere S^3 onto the two-dimensional sphere S^2. Introducing the symplectic form $\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \stackrel{\text{df}}{=} dc_5 \wedge dc_6 + dc_7 \wedge d(-c_8)$ we obtain $\{d_i, d_j\} = -2\sum_{k=1}^{3} \epsilon_{ijk} d_k$, $\{d_i, d_4\} = 0$ where ϵ_{ijk} is the completely antisymmetric tensor and $\{,\}$ is the Poisson bracket: $\{f_1, f_2\} = \sum_k (\frac{\partial f_1}{\partial p_k}\frac{\partial f_2}{\partial q_k} - \frac{\partial f_2}{\partial q_k}\frac{\partial f_1}{\partial p_k})$ (see also exercises 20.4, 20.6 in [4] for similar computations). Since X_1, X_2, X_3 are generators of $\mathfrak{su}(2)$ algebra it follows that the commutators $i[X_j, X_k]$ correspond to the Poisson brackets $\{d_j, d_k\}$, $j, k = 1, 2, 3$.

In the case of arbitrary j we obtain the same picture [15].

6. Discussion

We have constructed a class of ε-dependent classical mappings, arising from quantum maps, which for $\varepsilon = 0$ are linear (hence integrable) while for $\varepsilon \neq 0$ for some initial conditions they evolve like linear maps and for other initial conditions their evolution is nonlinear and even chaotic. This recalls the KAM theorem. The KAM theorem describes the fate of invariant tori of an integrable Hamiltonian system subject to a Hamiltonian perturbation [7–9]. If a perturbation is small enough then though some tori are destroyed the motion is confined to tori for most initial data. Moreover, for growing perturbation more tori are destroyed, giving rise to chaotic motion.

There are however several important differences. Most importantly, the maps in our case are not in general area preserving since they have attractors and repellers. Furthermore, we know the exact form of the linear manifold of the initial conditions on which the motion is exactly linear (the motion on the manifold can be stable or unstable) and this linear manifold persists for arbitrary ε.

Our results cast some light on the problem of geometric quantization—a geometric relation between quantum maps (8) and classical maps (9) arising from quantum kicked top dynamics [3], cf section 5, was found.

Analogous results can be obtained for other groups, e.g. $SU(p, q)$ [15].

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